Prediction of the behaviour of an unreliable system : application to the choice of the optimal maintenance policy

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Prédiction du comportement de systèmes non fiables. Application au choix de la politique de maintenance optimale.

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Résumé. - *Nous observons un système dont l'état est caractérisé par une variable unique. Nous supposons que l'évolution de l'état du système d'un instant à l'autre est régie par une distribution de probabilité exponentielle. Le système tombe en panne si son état dépasse une limite donnée. Dans ce cas, il doit être réparé, ce qui entraîne un coût de réparation. Il est aussi possible d'effectuer une maintenance d'un coût moindre avant la panne du système. Notre but est de déterminer une valeur d'alerte au-delà de laquelle une maintenance préventive sera réalisée si le système n'est pas en panne. Le critère est de minimiser le coût moyen de cette politique de maintenance.*

Mots-Clefs : Détection Précoce de Panne, Politique de Maintenance, Processus de Vieillissement, Réparation Préventive, Systèmes non Fiables.

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Prediction of the behaviour of an unreliable system. Application to the choice of the optimal maintenance policy

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Abstract. - *We consider a system the state of which is represented by the value of a unique variable. The state of the system can be observed at regular intervals of time. It evolves from one instant to the next one according to an exponential distribution. The system breakdowns if the state exceeds a given limit. In this case, it must be repaired, which results to a repair cost. It is also possible to maintain the system, i.e. to inspect it and, if required, adjust it before it breaks down. This results in a maintenance cost usually much smaller than the repair cost. We aim at defining a warning-value less than the breakdown limit such that, if the state lays between the warning value and the breakdown limit, a maintenance operation is performed to the system. The goal is to minimize the average cost.*

Keywords : Ageeing Process, Control Limit, Early Fault Detection, Maintenance Policy, Preventive Repair, Unreliable System.

1. Introduction

In this paper, we propose a predictive maintenance policy for an unreliable single-unit system.

Two different methods have been used by researchers to address this kind of problem :
The first one, known as the early detection of failure, is based on the analysis of the state of the system to detect when it becomes abnormal. BRUNET et al. [5] give a review of this approach. BASSEVILLE [3] [4] addresses a more specific problem of the detection of changes in the

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signals from the system. PATTON et al. [8] extend this approach to the diagnostic of the failure. In all cases, the early detection of failure implies the use of an analytical or statistical (LYONNET [6]) or cognitive model of the system.
In a second method, an optimal preventive maintenance policy is defined using the stochastic behaviour of the system. BARLOW and PROSCHAN [1] and more recently VALDEZ-FLOREZ and FELDMAN [11] give a survey on the research in this field. They classify these types of approach into three main categories : the inspection models where the problem is to schedule inspections (NAKAGAWA [7]), the minimal repair models that deal with the choice between the replacement of the whole system or just a part of it (BARLOW and HUNTER [2], SHEU et al. [9]), and the shock models for which a limit control has to be defined (ZUCKERMAN [12]).

In this paper, we propose a new approach to the problem that combines an early detection of failure and the choice of the optimal maintenance policy of a shock model. The stochastic behaviour of the system, the repair cost and the maintenance cost are used to compute an optimal warning-value that will trigger the predictive repair of the system.

In section 2, we introduce the model and calculate the distribution of the failure instant. In section 3, we present the maintenance policy and show how to find the optimal warning-value y*. The results are analyzed in section 4.

2. Model and Distribution of the failure instant

2.1 Model

We consider an unreliable system the state of which is defined by a unique parameter x. Let x_t be the state of the system at instant t. The transition probability from state x_{t-1} to state x_t only depends on state x_{t-1} and is given by a density of probability denoted by $f(x_t/x_{t-1})$. In this article, we consider the following density which depends on parameter μ :

$$f(x_t / x_{t-1}) = \begin{cases} \mu e^{-\mu(x_t - x_{t-1})} & \text{if } x_t \geq x_{t-1} \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

Since x never decreases, this process is an ageing process. The mean speed at which the value of x increases depends on the value of μ : the smaller μ, the greater this speed.

We assume that the system is in working order if $x \in [0, L), L > 0$. The system breaks down as soon as x becomes greater than or equal to L.

2.2 Distribution of the failure instant

The transition probability from state $x_{t-1}<L$ at time t-1 to state $x_t \geq L$ at time t is :

$$P(x_t \geq L / x_{t-1} < L) = \int_L^{+\infty} f(x_t / x_{t-1}) dx_t = \int_L^{+\infty} \mu e^{-\mu(x_t - x_{t-1})} dx_t$$

thus,

$$P(x_t \geq L / x_{t-1} < L) = e^{-\mu(L - x_{t-1})} \qquad (2)$$

More generally, the probability to exceed the limit L after t instants starting from the initial state $x_0<L$ is :

$$\begin{aligned}
&P(x_t \geq L \text{ and } x_{t-j} < L, \forall 1 \leq j < t / x_0 < L) \\
&= \int_{x_0}^{L} \int_{x_1}^{L} \dots \int_{x_{t-2}}^{L} \int_{L}^{+\infty} \prod_{j=1}^{t} f(x_j / x_{j-1}) dx_t dx_{t-1} \dots dx_2 dx_1 \\
&= \int_{x_0}^{L} \int_{x_1}^{L} \dots \int_{x_{t-2}}^{L} \int_{L}^{+\infty} \mu^t e^{-\mu(x_t - x_0)} dx_t dx_{t-1} \dots dx_2 dx_1 \\
&= \mu^{t-1} e^{-\mu(L - x_0)} \int_{x_0}^{L} \int_{x_1}^{L} \dots \int_{x_{t-2}}^{L} 1 \, dx_{t-1} \dots dx_2 dx_1
\end{aligned}$$

which leads to :

$$P(x_t \geq L \text{ and } x_{t-j} < L, \forall 1 \leq j < t / x_0 < L) = \frac{[\mu(L - x_0)]^{t-1}}{(t-1)!} e^{-\mu(L - x_0)} \qquad (3)$$

If we denote this probability by $\varphi(t)$ and set $\lambda = \mu(L - x_0)$, formula (3) can be rewritten as :

$$\varphi(t) = \frac{\lambda^{t-1}}{(t-1)!} e^{-\lambda} \qquad \text{for } t=1,2,\ldots,+\infty \tag{4}$$

The mean value corresponding to this distribution is :

$$E[t] = \sum_{j=0}^{t-1} j\varphi(j) = \sum_{j=0}^{t-1} j \frac{\lambda^{j-1}}{(j-1)!} e^{-\lambda} = \lambda + 1,$$

and the variance is :

$$\sigma^2 = E[t^2] - (E[t])^2 = \sum_{j=0}^{t-1} j^2 \frac{\lambda^{j-1}}{(j-1)!} e^{-\lambda} - (\lambda+1)^2 = \lambda .$$

Thus, following result holds :

Result 1 : The probability to reach the failure mode after t instants follows a Poisson-like distribution whose mean value and variance are respectively $\lambda+1$ and λ.

3. Maintenance policy

Let us consider the following maintenance policy :

- if the system breaks down (i.e. if its state x_t exceeds the limit L), it is repaired. The cost of this repair is CF.

- if the current state x_t of the system exceeds a given warning-value y<L but remains less than L, a maintenance operation is done on the system. The cost of this preventive maintenance is CP<CF.

- After being repaired, the state of the system is re-initialized at $x_0=0$. Note that if $x_0>0$, it is always possible to return to the previous assumption by setting $x=x-x_0$ and $L=L-x_0$.

Figure 1 shows a possible sequence of states.

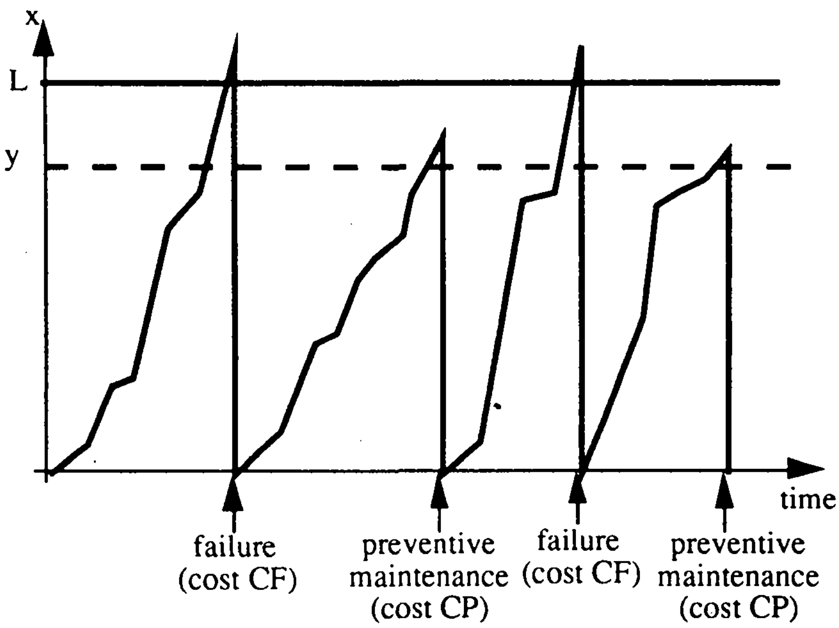

Fig. 1 : Evolution of the state of the system

We are looking for the maintenance policy (i.e. for the optimal warning-value y*) which guarantees the minimal average cost.

Let α be the ratio between CF and CP ($\alpha \geq 1$). We are looking for y* which minimizes the mean maintenance cost function g :

$$g(y) = P(y \leq x_t < L \text{ and } x_{t-1} < y) + \alpha P(x_t \geq L \text{ and } x_{t-1} < y) \quad (5)$$

where the first term of the right side of (5) represents the mean cost of preventive maintenance and the second term corresponds to the mean cost of failure repair. This formulation implies that a stationary behaviour of the system exists. We prove in appendix that this assumption holds.

3.1 Probability to be greater than or less than the warning-value y

Let us denote by p(x) the density of probability for the system to be in the elementary interval [x, x+dx) at any instant t :

$$p(x)dx = P(x \leq X_t < x + dx)$$

We assume that the system reachs a steady state.

According to relation (1), the probability to be in state x at instant t depends on the state of the system at instant t-1. Thus :

$$P(x \leq X_t < x + dx) = P(x \leq X_t < x + dx \wedge (X_{t-1} < y \vee X_{t-1} \geq y))$$
$$= P(x \leq X_t < x + dx \wedge X_{t-1} < y) \vee P(x \leq X_t < x + dx \wedge X_{t-1} \geq y) \quad (6)$$

where stands $\wedge$ for "and" and $\vee$ for "or".

The first part of the right side of this equality represents the probability to reach state x starting from $X_{t-1}<y$. We know (see (1)) that X_{t-1} is smaller or equal to x . Thus, we can write :

$$P(x \leq X_t < x + dx \wedge X_{t-1} < y)$$
$$= \int_{0^+}^{Min(x,y)} P(x \leq X_t < x + dx \wedge z \leq X_{t-1} < z + dz)dz$$
$$= \int_{0^+}^{Min(x,y)} P(x \leq X_t < x + dx / X_{t-1} = z)P(z \leq X_{t-1} < z + dz)dz \quad (7)$$

The second term of the right side of equation (6) concerns the evolution which follows a breakdown. The state of the system being initialized at $x_0=0$ immediately after each breakdown, the second term can be written as :

$$P(x \leq X_t < x + dx \wedge X_{t-1} \geq y)$$
$$= \int_{y}^{+\infty} P(x \leq X_t < x + dx \wedge z \leq X_{t-1} < z + dz)dz$$
$$= \int_{y}^{+\infty} P(x \leq X_t < x + dx / X_{t-1} = 0)P(z \leq X_{t-1} < z + dz)dz \quad (8)$$

Using equality (7) and (8), relation (6) can be rewritten as :

$$p(x)dx = P(x \leq X_t < x + dx)$$
$$= \int_{0^+}^{Min(x,y)} P(x \leq X_t < x + dx / X_{t-1} = z)P(z \leq X_{t-1} < z + dz)dz$$
$$+ \int_{y}^{+\infty} P(x \leq X_t < x + dx / X_{t-1} = 0)P(z \leq X_{t-1} < z + dz)dz$$

If we use the density of probability defined in equations (1) , we obtain :

$$p(x) = \int_{0^+}^{\mathrm{Min}(x,y)} f(x/z)p(z)\,dz + \int_{y}^{+\infty} f(x/0)p(z)\,dz$$
$$= \int_{0^+}^{\mathrm{Min}(x,y)} \mu e^{-\mu(x-z)} p(z)\,dz + \int_{y}^{+\infty} \mu e^{-\mu x} p(z)\,dz \qquad (9)$$

The derivative of p for x<y, is :

$$p'(x) = \frac{d}{dx}\left[\int_{0^+}^{x} \mu e^{-\mu(x-z)} p(z)\,dz + \int_{y}^{+\infty} \mu e^{-\mu x} p(z)\,dz\right]$$

$$= -\mu \int_{0^+}^{x} \mu e^{-\mu(x-z)} p(z)\,dz + \mu p(x) - \mu \int_{y}^{+\infty} \mu e^{-\mu x} p(z)\,dz$$
$$= \mu p(x) - \mu p(x) = 0$$

p'(x) is equal to zero for x<y. Thus p(x) is equal to a constant for x<y. Let us denote this constant by C.

The density p for $x \geq y$ is then :

$$p(x) = \int_{0^+}^{y} f(x/z)p(z)\,dz + \int_{y}^{+\infty} f(x/0)p(z)\,dz$$
$$= \int_{0^+}^{y} \mu e^{-\mu(x-z)} C\,dz + \mu e^{-\mu x} \int_{y}^{+\infty} p(z)\,dz$$

Let $P_0 = \int_{y}^{+\infty} p(z)\,dz$. P_0 is the probability for the state of the system to exceed y.

With this notation, for $x \geq y$, p(x) can be rewritten as:

$$p(x) = \int_{0^+}^{y} \mu e^{-\mu(x-z)} C\,dz + \mu e^{-\mu x} P_0$$
$$= \mu e^{-\mu x}\left(\frac{C}{\mu}(e^{\mu y} - 1) + P_0\right)$$

By replacing p(x) by this expression in the expression of P_0, we obtain :

$$P_0 = \frac{C}{\mu}$$

Moreover, we know that :

$$P(0 < X_t < y) + P(X_t \geq y) = 1$$

Thus :

$$\int_{0^+}^{y} C dz + P_0 = 1 \qquad \text{which leads to} \quad C = \frac{\mu}{1+\mu y} \quad \text{and} \quad P_0 = \frac{1}{1+\mu y}$$

We finally obtain the density of probability of x :

$$p(x) = \begin{cases} \dfrac{\mu}{1+\mu y} & \text{for } x < y \\ \dfrac{\mu e^{-\mu(x-y)}}{1+\mu y} & \text{for } x \geq y \end{cases}$$

3.2 Cost function

We are now able to calculate the two terms of the cost function (5) :

$$\begin{aligned} P(x_t \geq L \text{ and } x_{t-1} < y) &= \int_L^{+\infty} \left(\mu e^{-\mu x_t} P_0 + \int_{0^+}^{y} C \mu e^{-\mu(x_t - x_{t-1})} dx_{t-1} \right) dx_t \\ &= \frac{e^{-\mu(L-y)}}{1+\mu y} \end{aligned}$$

and

$$\begin{aligned} P(y \leq x_t < L \text{ and } x_{t-1} < y) &= P(x_t \geq y \text{ and } x_{t-1} < y) - P(x_t \geq L \text{ and } x_{t-1} < y) \\ &= P_0 - P(x_t \geq L \text{ and } x_{t-1} < y) \\ &= \frac{1 - e^{-\mu(L-y)}}{1+\mu y} \end{aligned}$$

Thus, the cost function g(y) becomes :

$$g(y) = \frac{e^{-\mu(L-y)}(\alpha - 1) + 1}{1+\mu y} \tag{10}$$

A similar model has been studied by Taylor [10] using a different approach.

3.3 Optimal warning-value y*

We are looking for y* that verifies :

$$g(y^*) = \min_{y \in (0,L)} g(y)$$

In that case, y* is one of the solutions of $g'(y) = 0$ where g' is the derivative function of g if $y^* \in (0,L)$. Otherwise, y* is equal either to 0 or to L.

Setting $A = (\alpha - 1)e^{-\mu L}$, equality (10) becomes :

$$g(y) = \frac{Ae^{\mu y} + 1}{1 + \mu y}$$

and then,

$$g'(y) = \frac{\mu^2 Aye^{\mu y} - \mu}{(1 + \mu y)^2} \qquad (11)$$

The equation g'(y*)=0, is equivalent to :

$$y^* e^{\mu y^*} = \frac{1}{\mu A} \qquad (12)$$

Thus, result 2 holds.

Result 2 : the optimal warning-value y* is solution of equation (12).

Unfortunatly, this equation has no analytical solution.

Let us study the behaviour of g(y). Its second derivative is :

$$g''(y) = \frac{\mu^2 (2 + Ae^{\mu y} + A\mu^2 y^2 e^{\mu y})}{(1 + \mu y)^3}$$

The second derivative of g is strictly positive. The cost function g is then convex. It has a unique minimum at point y*.

We know that y* belongs to (0,L). We can then find the value of y* using any appropriate numerical method (dichotomy for instance). We can also try to bound y* using equation (12) :

By replacing μy^* by 0, we obtain : $y^* < \frac{1}{\mu A}$, By setting $\mu y^*=\mu L$, we obtain : $y^* > \frac{e^{-\mu L}}{\mu A}$,

Finally, setting $y^*=L$, we obtain : $y^* > -\frac{\log(\mu LA)}{\mu}$

Thus, we can use the following bounds for a more efficient computation of y* using dichotomy:

$$\mathrm{Min}\left(\mathrm{Max}\left(-\frac{\log(\mu LA)}{\mu}, \frac{e^{-\mu L}}{\mu A}\right), L\right) \leq y^* \leq \mathrm{Min}\left(\frac{1}{\mu A}, L\right)$$

4. Analysis of the results

The cost function g depends on y but also on A.

For a large A (A>>1), y* tends to 0. This comes from the fact that if A is large, either the cost of an after-failure repair is much more important than the cost of a preventive repair (α large), and/or the ageing process is very quick (μ large), and/or the failure limit L is very small. The best maintenance policy consists of performing a preventive maintenance at each instant.

If A becomes smaller than 1, the optimal warning-value y* increases toward limit L. This means that the cost of an after-failure repair is close to the cost of a preventive repair (α -> 1), and/or the system evolves slowly (μ is small), and/or the limit L is high.

For A=0, we have :

$$g(y) = \frac{1}{1+\mu y}$$

This function is strictly decreasing. Its minimum is at infinity. When the cost of an after-failure repair is the same as the cost of a preventive repair (A=0), the optimal policy is to wait until the system breakdowns.
In Figure 2, we show the evolution of y* with regard to α and μ for L=10. When the normalized cost α increases, the failure repairs have a more important effect on the policy cost than the preventive maintenance, and thus y* decreases. When μ is small, y* decreases quickly toward 0

when α increases. Indeed, the system evolves quickly and thus the probability to fail after a small period is high. Moreover, for a given α, the higher μ (the slower ageeing process), the higher y*.

5. Conclusion

We described an unreliable system whose state evolution from one instant to the next follows a negative exponential distribution.We shown that the probability distribution of the breakdown instant follows a Poisson distribution.

Thus, an optimal maintenance policy can be found, considering the repair cost, the preventive maintenance cost and the parameters of the system.

The stochastic model considered in this paper is quite restrictive, but it constitutes a basis for further research. In particular, it should be interesting to consider other distributions for the evolution from one state to another as well as the generalization of this model to a multi-state model.

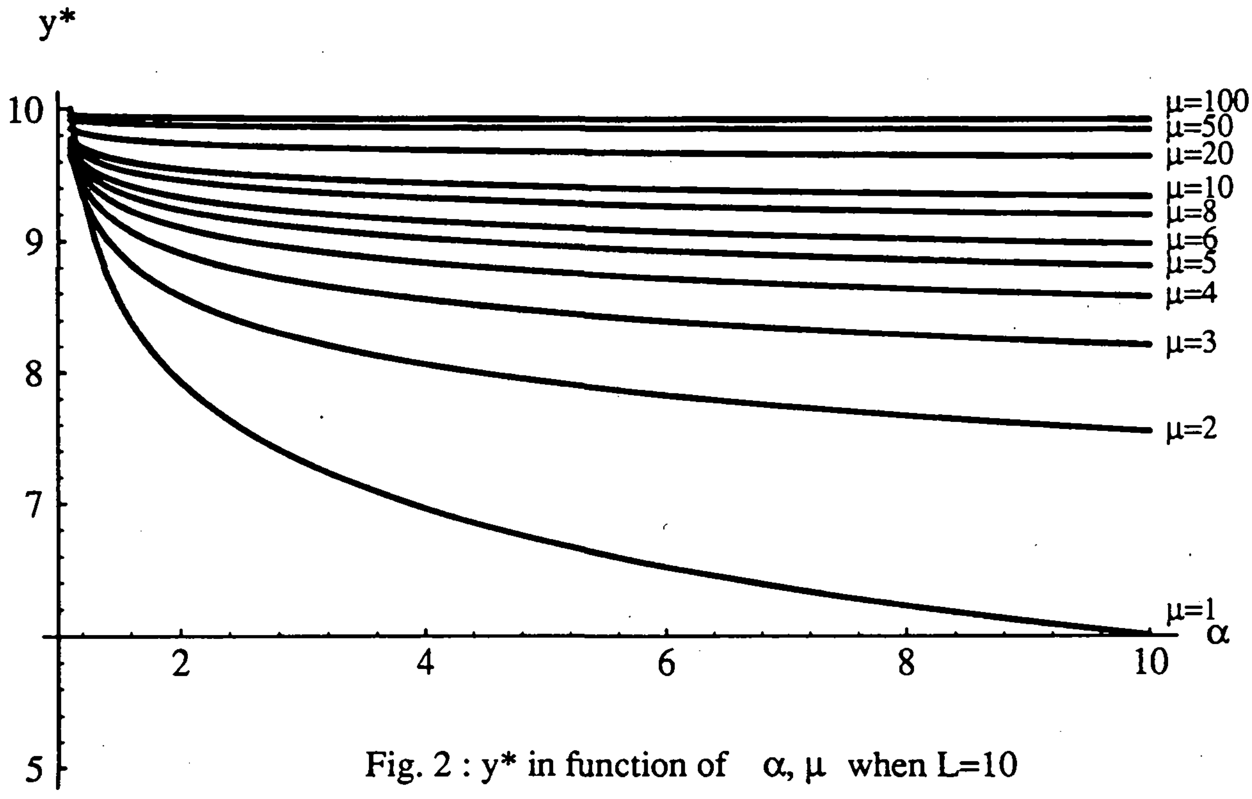

Fig. 2 : y* in function of α, μ when L=10

Appendix : Proof of the formulation of the cost function

The long-run expected cost can be defined as $\lim_{T\to+\infty} \frac{E[C_T]}{T}$ where C_T is the total maintenance cost over period T. According to Stolz's theorem, if $\lim_{T\to+\infty} \left(E[C_T]-E[C_{T-1}]\right)$ exists, then:

$$\lim_{T\to+\infty} \frac{E[C_T]}{T} = \lim_{T\to+\infty} \left(E[C_T]-E[C_{T-1}]\right).$$

We thus focus on the computation of $E[C_T]$.

The following notations will be used:

$p_1(s)$: probability that the system is at work for s units of time since the last repair,

$p_2(s)$: probability that state x reaches a value belonging to [y,L), s units of time after a repair,

$p_3(s)$: probability that state x becomes greater than L, s units of time after a repair.

These probabilities can be expressed as:

$$p_1(s) = \sum_{i=s}^{+\infty} \frac{(\mu y)^s}{s!} e^{-\mu y}$$

$$p_2(s) = \frac{(\mu y)^{s-1}}{(s-1)!}\left[e^{-\mu y} - e^{-\mu L}\right]$$

$$p_3(s) = \frac{(\mu y)^{s-1}}{(s-1)!} e^{-\mu L}$$

Note that $p_1(0) = 1$.

Let us introduce a binary variable v_i defined as follows:

$$v_i = \begin{cases} 1 & \text{if the i}-\text{th repair is a preventive one} \\ 0 & \text{if the i}-\text{th repair is after failure} \end{cases}$$

Let t_i be the number of units of time between the ith and the (i-1)th repairs. If n denotes the number of repairs, the following equality holds:

$$E[C_T] = \sum_{n=1}^{T} \sum_{s=0}^{T-n} p_1(s) \sum_{v_1=0}^{1} \cdots \sum_{v_n=0}^{1} \sum_{t_1=1}^{T-s-n+1} \sum_{t_2=1}^{T-s-n-t_1+2} \cdots \sum_{t_{n-1}=1}^{T-s-1-\sum_{i=1}^{n-2} t_i}$$

$$\left\{ \sum_{j=1}^{n} \left[\left(1-v_j\right)\alpha + v_j \right] \right\} \left\{ \prod_{j=1}^{n-1} \left[\left(1-v_j\right) p_3\left(t_j\right) + v_j p_2\left(t_j\right) \right] \right\} \cdot$$

$$\left[(1-v_n) p_3\left(T - s - \sum_{i=1}^{n-1} t_i \right) + v_n p_2 \left(T - s - \sum_{i=1}^{n-1} t_i \right) \right]$$

Taking into account equality:

$$\left(1-v_j\right) p_3(t) + v_j p_2(t) = \frac{(\mu y)^{t_j - 1}}{(t_j - 1)!} \left[\left(1-v_j\right) e^{-\mu L} + v_j \left(e^{-\mu y} - e^{-\mu L} \right) \right]$$

we obtain:

$$E[C_T] = \sum_{n=1}^{T} \sum_{s=0}^{T-n} p_1(s) \sum_{v_1=0}^{1} \cdots \sum_{v_n=0}^{1} \left\{ \sum_{j=1}^{n} \left[\left(1-v_j\right)\alpha + v_j \right] \right\} \cdot$$

$$\left\{ \prod_{j=1}^{n} \left[\left(1-v_j\right) e^{-\mu L} + v_j \left(e^{-\mu y} - e^{-\mu L} \right) \right] \right\} \cdot$$

$$\left[\sum_{t_1=1}^{T-s-n+1} \sum_{t_2=1}^{T-s-n-t_1+2} \cdots \sum_{t_{n-1}=1}^{T-s-1-\sum_{i=1}^{n-2} t_i} \left(\prod_{j=1}^{n-1} \frac{(\mu y)^{t_j - 1}}{(t_j - 1)!} \right) \cdot \frac{(\mu y)^{T-s-1-\sum_{i=1}^{n-1} t_i}}{\left(T - s - 1 - \sum_{i=1}^{n-1} t_i \right)!} \right]$$

Consider the sum in the last pair of brackets and denote it by S. It can be re-written as:

$$S = (\mu y)^{T-s-n} \sum_{t_1=1}^{T-s-n+1} \sum_{t_2=1}^{T-s-n-t_1+2} \cdots$$

$$\sum_{t_{n-1}=1}^{T-s-1-\sum_{i=1}^{n-2} t_i} \left(\prod_{j=1}^{n-1} \frac{1}{(t_j-1)!} \right) \frac{1}{\left(T-s-1-\sum_{i=1}^{n-1} t_i \right)!}$$

$$= (\mu y)^{T-s-n} \sum_{t_1=1}^{T-s-n+1} \frac{1}{(t_1-1)!} \sum_{t_2=1}^{T-s-n-t_1+2} \frac{1}{(t_2-1)!} \cdots$$

$$\sum_{t_{n-1}=1}^{T-s-1-\sum_{i=1}^{n-2} t_i} \frac{1}{(t_{n-1}-1)!\left(T-s-1-\sum_{i=1}^{n-1} t_i \right)!}$$

Applying the following relation (n-1) times:

$$\sum_{i=1}^{k} \frac{m^{i-1}}{(i-1)!(k-i)!} = \frac{(m+1)^{k-1}}{(k-1)!}$$

we obtain:

$$S = \frac{(n\mu y)^{T-s-n}}{(T-s-n)!}$$

Thus $E[C_T]$ becomes:

$$E[C_T] = \sum_{n=1}^{T} \sum_{s=0}^{T-n} p_1(s) \frac{(n\mu y)^{T-s-n}}{(T-s-n)!} \sum_{v_1=0}^{1} \cdots$$

$$\sum_{v_n=0}^{1} \left\{ \sum_{j=1}^{n} \left[\left(1-v_j\right)\alpha + v_j \right] \right\} \prod_{j=1}^{n} \left[\left(1-v_j\right) e^{-\mu L} + v_j \left(e^{-\mu y} - e^{-\mu L} \right) \right]$$

Let k be the number of j indexes such that $v_j = 1$. The previous equality can be re-written as:

$$E[C_T] = \sum_{n=1}^{T} \sum_{s=0}^{T-n} p_1(s) \frac{(n\mu y)^{T-s-n}}{(T-s-n)!}$$

$$\sum_{k=0}^{n} \left[[k+(n-k)\alpha] C_n^k \left(e^{-\mu y} - e^{-\mu L} \right)^k \left(e^{-\mu L} \right)^{n-k} \right]$$

Considering relation:

$$\sum_{k=0}^{n}[k+(n-k)\alpha]C_n^k a^k b^{n-k} = n(a+\alpha b)(a+b)^{n-1}$$

we further obtain:

$$E[C_T]=\sum_{n=1}^{T}\sum_{s=0}^{T-n} p_1(s)\frac{(n\mu y)^{T-s-n}.n}{(T-s-n)!}e^{-(n-1)\mu y}\left[e^{-\mu y}+(\alpha-1)e^{-\mu L}\right]$$

$$=\left[e^{-\mu(L-y)}(\alpha-1)+1\right]\sum_{n=1}^{T}\sum_{s=0}^{T-n}\frac{(n\mu y)^{T-s-n}.n}{(T-s-n)!}e^{-n\mu y}p_1(s)$$

$$=\left[e^{-\mu(L-y)}(\alpha-1)+1\right]\sum_{n=1}^{T}\sum_{j=0}^{T-n}\frac{(n\mu y)^{j}.n}{j!}e^{-n\mu y}p_1(T-n-j) \quad (A1)$$

Therefore:

$$E[C_{T-1}]=\left[e^{-\mu(L-y)}(\alpha-1)+1\right]\sum_{n=1}^{T-1}\sum_{j=0}^{T-n-1}\frac{(n\mu y)^{j}.n}{j!}e^{-n\mu y}p_1(T-n-j-1)$$

$E[C_T]$ can also be expressed as follows, starting from the second right side of equality (A1):

$$E[C_T]=\left[e^{-\mu(L-y)}(\alpha-1)+1\right]\left\{\sum_{n=1}^{T-1}\sum_{j=0}^{T-n-1}\frac{(n\mu y)^{j}.n}{j!}e^{-n\mu y}p_1(T-n-j)\right.$$
$$\left.+\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}.n}{(T-n)!}p_1(0)\right\}$$

Hence:

$$E[C_T]-E[C_{T-1}]=\left[(\alpha-1)e^{-\mu(L-y)}+1\right]\left\{\sum_{n=1}^{T-1}\sum_{j=0}^{T-n-1}\frac{(n\mu y)^{j}.n}{j!}e^{-n\mu y}\right.$$
$$\left[p_1(T-n-j)-p_1(T-n-j-1)\right]$$
$$\left.+\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}.n}{(T-n)!}p_1(0)\,e^{-n\mu y}\right\}$$
$$=\left[(\alpha-1)e^{-\mu(L-y)}+1\right]\left\{-\sum_{n=1}^{T-1}\sum_{j=0}^{T-n-1}\frac{(n\mu y)^{j}.n}{j!}e^{-(n+1)\mu y}.\right.$$
$$\left.\frac{(\mu y)^{T-n-1-j}}{(T-n-1-j)!}+\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}.n}{(T-n)!}e^{-n\mu y}\right\}$$

$$= \left[(\alpha-1)\,e^{-\mu(L-y)}+1\right]\left\{+\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}.n}{(T-n)!}\,e^{-n\mu y}\right.$$

$$\left.-\sum_{n=1}^{T-1}\frac{[(n+1)\mu y)]^{T-n-1}.n}{(T-n-1)!}\,e^{-(n+1)\mu y}\right\}$$

$$= \left[(\alpha-1)\,e^{-\mu(L-y)}+1\right]\left\{+\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}.n}{(T-n)!}\,e^{-n\mu y}\right.$$

$$\left.-\sum_{j=1}^{T}\frac{(j-1)(j\mu y)^{T-j}}{(T-j)!}\,e^{-j\mu y}\right\}$$

$$= \left[(\alpha-1)\,e^{-\mu(L-y)}+1\right]\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}}{(T-n)!}\,e^{-n\mu y}$$

We now have to compute:

$$\lim_{T\to+\infty}\left(E[C_T]-E[C_{T-1}]\right)=\left[(\alpha-1)\,e^{-\mu(L-y)}+1\right].$$

$$\lim_{T\to+\infty}\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}}{(T-n)!}\,e^{-n\mu y}$$

Let us prove that:

$$\lim_{T\to+\infty}\sum_{n=1}^{T}\frac{(n\mu y)^{T-n}}{(T-n)!}\,e^{-n\mu y}=\frac{1}{1+\mu y}.$$

For this purpose, consider a complex function:

$$f(z)=\frac{1}{1-z\,e^{\varphi(z-1)}}-\frac{1}{(1-z)(1+\varphi)}$$

where z is a complex variable and φ a positive real number. f(z) can be rewritten as:

$$f(z)=\frac{\varphi(1-z)+z\left(e^{\varphi(z-1)}-1\right)}{\left(1-z\,e^{\varphi(z-1)}\right)(1-z)(1+\varphi)}$$

We will show that f(z) is a holomorphic function on a disc of radius strictly bigger than 1. To do so, we first show that f(z) has a unique pole z = 1 on a unit disc and we then show that this pole is removable.

To find the poles of f(z), we have to solve the following equation:
$$\left(1 - z\, e^{\varphi(z-1)}\right)(1 - z) = 0$$

One solution is z = 1. Other solutions are solutions of:
$$\left(1 - z\, e^{\varphi(z-1)}\right) = 0 \qquad \text{(A2)}$$

We want to show that any solution to equation (A2) is such that |z| > 1 or z = 1.

Suppose that there is a solution such that |z| ≤ 1 and z ≠ 1. Then we should have:
$$\left|z\, e^{\varphi(z-1)}\right| = 1$$

Let z = a + ib. In that case:
$$\left|z\, e^{\varphi(z-1)}\right| = \sqrt{a^2 + b^2}\; e^{\varphi(a-1)}$$

Since |z| ≤ 1, $a^2 + b^2 \leq 1$ and:
$$\left|z\, e^{\varphi(z-1)}\right| \leq e^{\varphi(a-1)}$$

But, since |z| ≤ 1 and z ≠ 1, a < 1 and:
$$e^{\varphi(a-1)} < 1$$

Finally:
$$\left|z\, e^{\varphi(z-1)}\right| < 1$$

This is in contradiction with the assumption that:
$$\left|z\, e^{\varphi(z-1)}\right| = 1$$

Up to now, we showed that f(z) has a unique pole z = 1 on the closed unit disc. We show now that this pole is removable. Using L'Hospital's law, we have:

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} \frac{-\varphi + e^{\varphi(z-1)} - 1 + \varphi z\, e^{\varphi(z-1)}}{(1+\varphi)\left[z\, e^{\varphi(z-1)} - 1 - e^{\varphi(z-1)}(1+\varphi z)(1-z)\right]}$$

$$= \lim_{z\to 1} \frac{e^{\varphi(z-1)}(2+\varphi z)}{(1+\varphi)\left[2\, e^{\varphi(z-1)}(1+\varphi z) - \varphi e^{\varphi(z-1)}(2+\varphi z)(1-z)\right]}$$

$$= \frac{2+\varphi}{2(1+\varphi)^2}$$

This ends up the proof that f(z) is holomorphic on a disc of radius strictly bigger than 1. On this disc, f(z) can be expanded as a Taylor series which converges when z = 1. Note that, for any $n \geq 1$:

$$f^{(n)}(z) = \frac{1}{\left(1 - z\, e^{\varphi(z-1)}\right)^{n+1}} \sum_{i=1}^{n} \varphi^{n-i} e^{i\varphi(z-1)} \sum_{j=0}^{i} (yz)^{j} \left[C_n^{i-j} \sum_{k=0}^{j} (i-k)^{n+j-i} \frac{(i-k)!}{(j-k)!} C_{n+1}^{k} (-1)^k \right] - \frac{n!}{(1-z)^{n+1}(1+\varphi)}$$

Thus:

$$f^{(n)}(0) = \sum_{i=1}^{n} \varphi^{n-i} e^{-i\varphi}\, C_n^i\, i^{n-i}\, i! - \frac{n!}{1+\varphi}$$

$$= n!\left[\sum_{i=1}^{n} \frac{(i\varphi)^{n-i}}{(n-i)!} e^{-i\varphi} - \frac{1}{1+\varphi}\right]$$

Since Taylor expansion of f(z) converges at z = 1, we obtain:

$$\lim_{n\to+\infty} \left|\frac{f^{(n)}(0)}{n!}\right| = \lim_{n\to+\infty} \left|\sum_{i=1}^{n} \frac{(i\varphi)^{n-i}}{(n-i)!} e^{-i\varphi} - \frac{1}{1+\varphi}\right| = 0$$

or equivalently:

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \frac{(i\varphi)^{n-i}}{(n-i)!} e^{-i\varphi} = \frac{1}{1+\varphi}$$

This leads to:

$$\lim_{T \to +\infty} \sum_{n=1}^{T} \frac{(n\mu y)^{T-n}}{(T-n)!} e^{-n\mu y} = \frac{1}{1+\mu y}$$

and:

$$\lim_{T \to +\infty} \frac{E[C_T]}{T} = \frac{(\alpha - 1)\, e^{-\mu(L-y)} + 1}{1 + \mu y}$$

We obtain the same result as using the steady state concept given in (10).

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